ELEN E3401: Electromagnetics

Spring 2025

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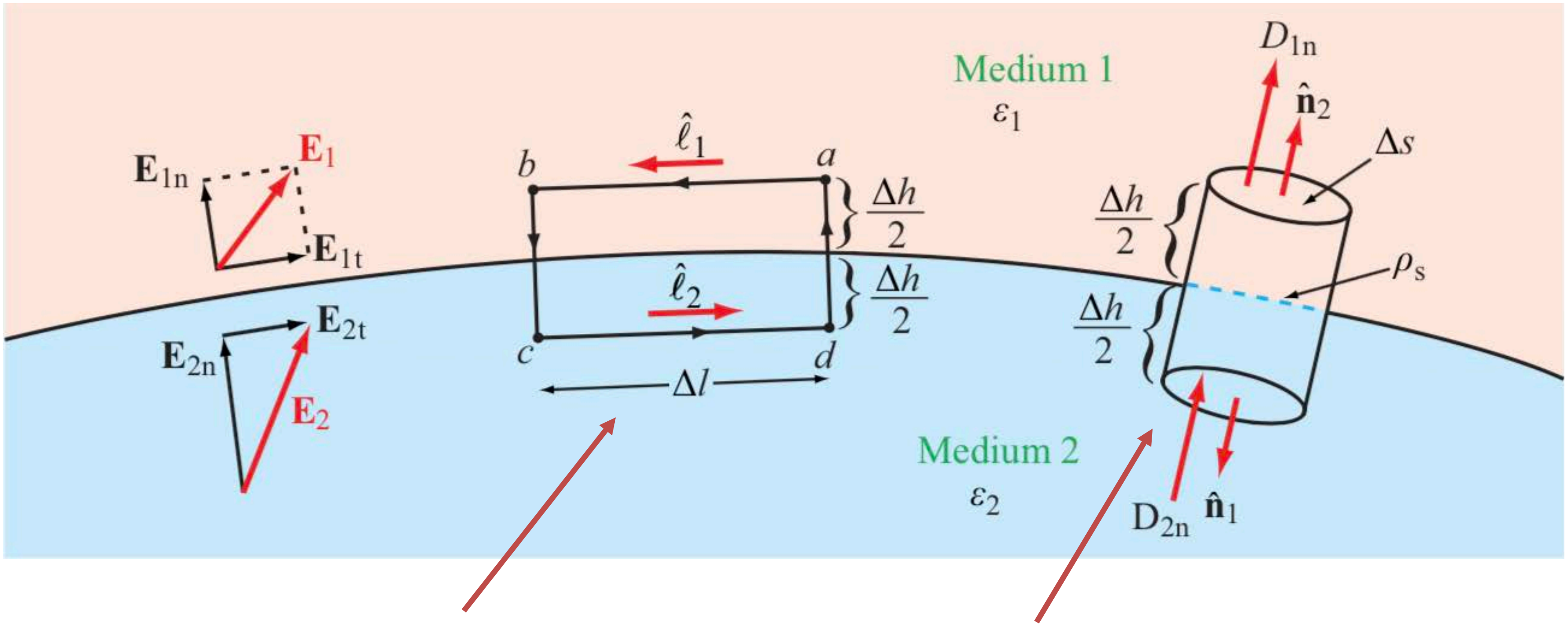
Lecture #13

COLUMBIA | ENGINEERING
The Fu Foundation School of Engineering and Applied Science

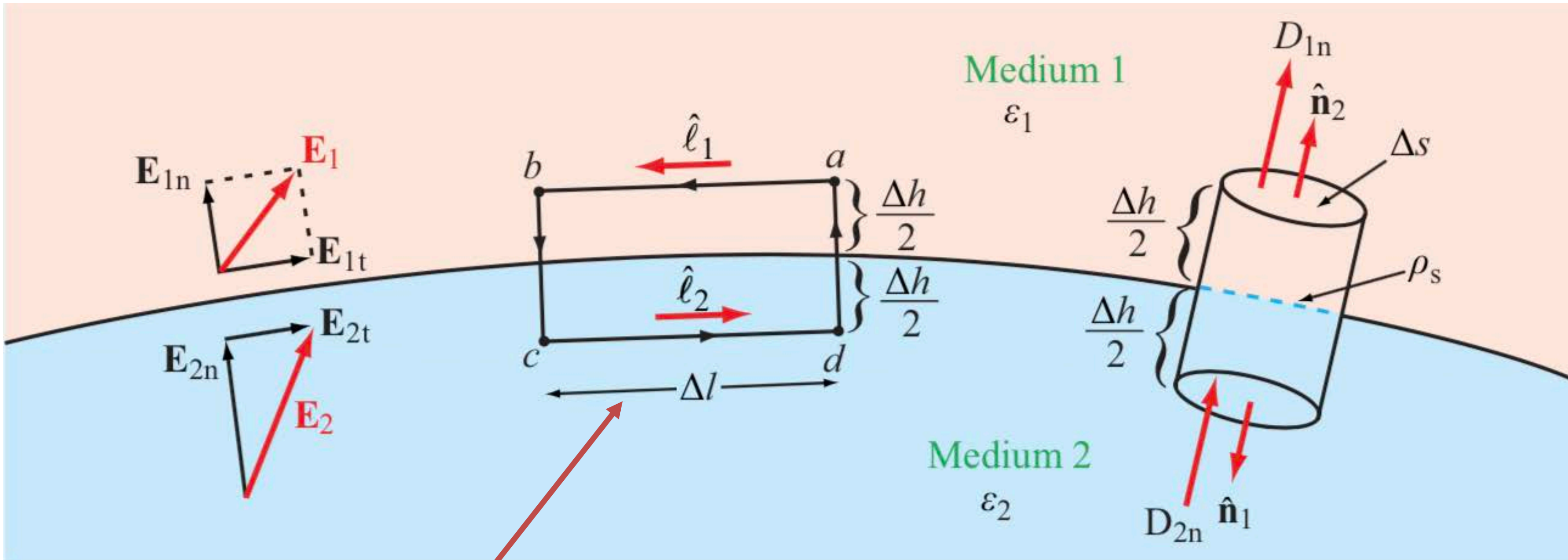
Tangential/normal components of $\vec{E}$ and $\vec{D}$



Faraday: $\vec{\nabla} \times \vec{E} = 0 \rightarrow \oint_C \vec{E} \cdot d\vec{l} = 0$

Gauss: $\vec{\nabla} \cdot \vec{D} = \rho_V \rightarrow \oint_S \vec{D} \cdot d\vec{s} = Q$

Tangential components of $\vec{E}$ and $\vec{D}$



Consider closed rectangular loop **a-b-c-d-a**

Conservative field: $\oint_C \vec{E} \cdot d\vec{l} = 0$ Line integral around closed path is zero

Let $\Delta h \to 0$ then $\overline{bc}$ and $\overline{da} \to 0$

$$\oint_C \vec{E} \cdot d\vec{l} = \int_a^b \underbrace{\vec{E}_1}_{\text{Medium 1}} \cdot \hat{l}_1 dl + \int_c^d \underbrace{\vec{E}_2}_{\text{Medium 2}} \cdot \hat{l}_2 dl = 0$$

Tangential components of $\vec{E}$ and $\vec{D}$

$$\left.\begin{aligned}\vec{E}_1 &= \vec{E}_{1t} + \vec{E}_{1n} \\ \vec{E}_2 &= \vec{E}_{2t} + \vec{E}_{2n}\end{aligned}\right\}$$ Tangential and normal components to the boundary

$$\hat{l}_1 = -\hat{l}_2 \qquad (\vec{E}_1 - \vec{E}_2) \cdot \hat{l}_1 = 0$$

To satisfy Faraday's law (closed contour, conservative field):
Component of $\vec{E}_1$ along $\hat{l}_1$ must equal component of $\vec{E}_2$ along $\hat{l}_1$ for all $\hat{l}_1$ tangential to the boundary

$$\boxed{\vec{E}_{1t} = \vec{E}_{2t}} \qquad [\mathrm{V/m}]$$

Tangential components of $\vec{E}$ and $\vec{D}$

Tangential component of the $\vec{E}$ field is <u>continuous</u> across boundary for <u>any</u> 2 media

$$\vec{D}_1 = \epsilon_1 \vec{E}_{1t} + \epsilon_1 \vec{E}_{1n}$$

$$\vec{D}_2 = \epsilon_2 \vec{E}_{2t} + \epsilon_2 \vec{E}_{2n}$$

$$\Rightarrow$$

$$\vec{D}_{1t} = \epsilon_1 \vec{E}_{1t}$$

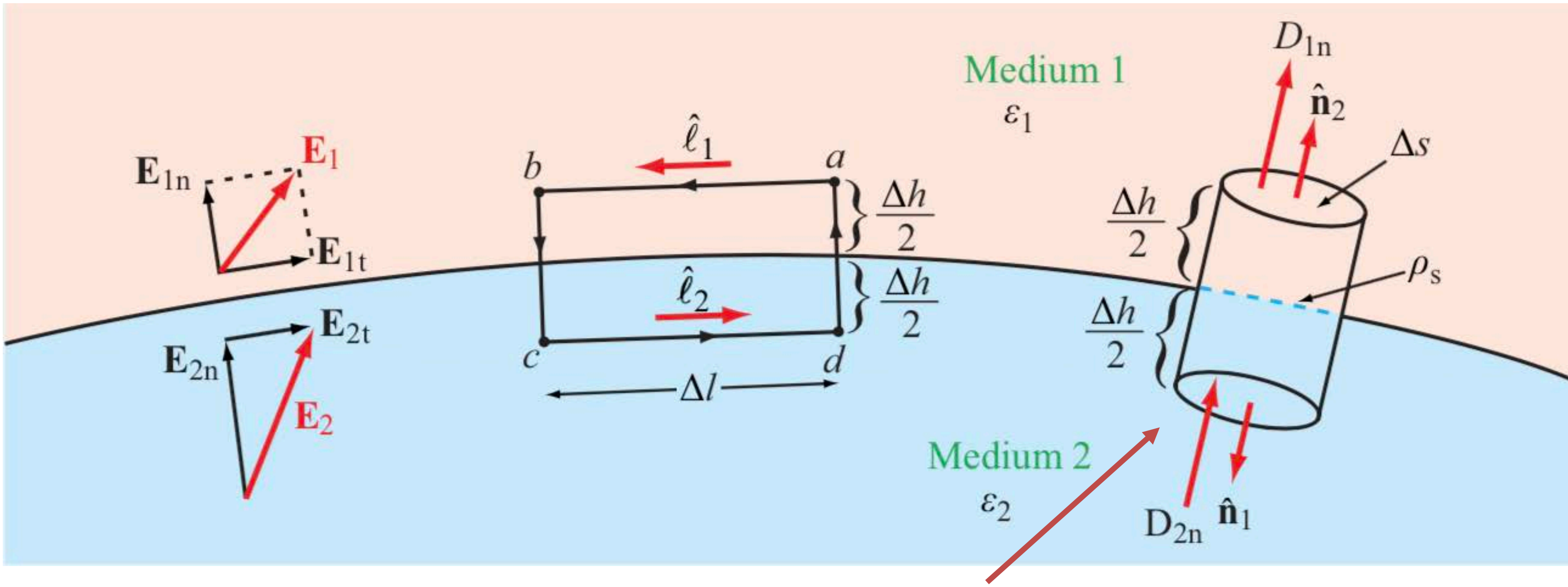
$$\vec{D}_{2t} = \epsilon_2 \vec{E}_{2t}$$

$$\vec{D}_{2t} = \epsilon_2 \vec{E}_{1t}$$

(since $\vec{E}_{1t} = \vec{E}_{2t}$)

$$\frac{\vec{D}_{1t}}{\epsilon_1} = \frac{\vec{D}_{2t}}{\epsilon_2} \qquad \frac{\vec{D}_{1t}}{\vec{D}_{2t}} = \frac{\epsilon_1}{\epsilon_2}$$

Normal components of $\vec{E}$ and $\vec{D}$



Apply Gauss's Law: total outward flux through cylinder must equal total charge inside.

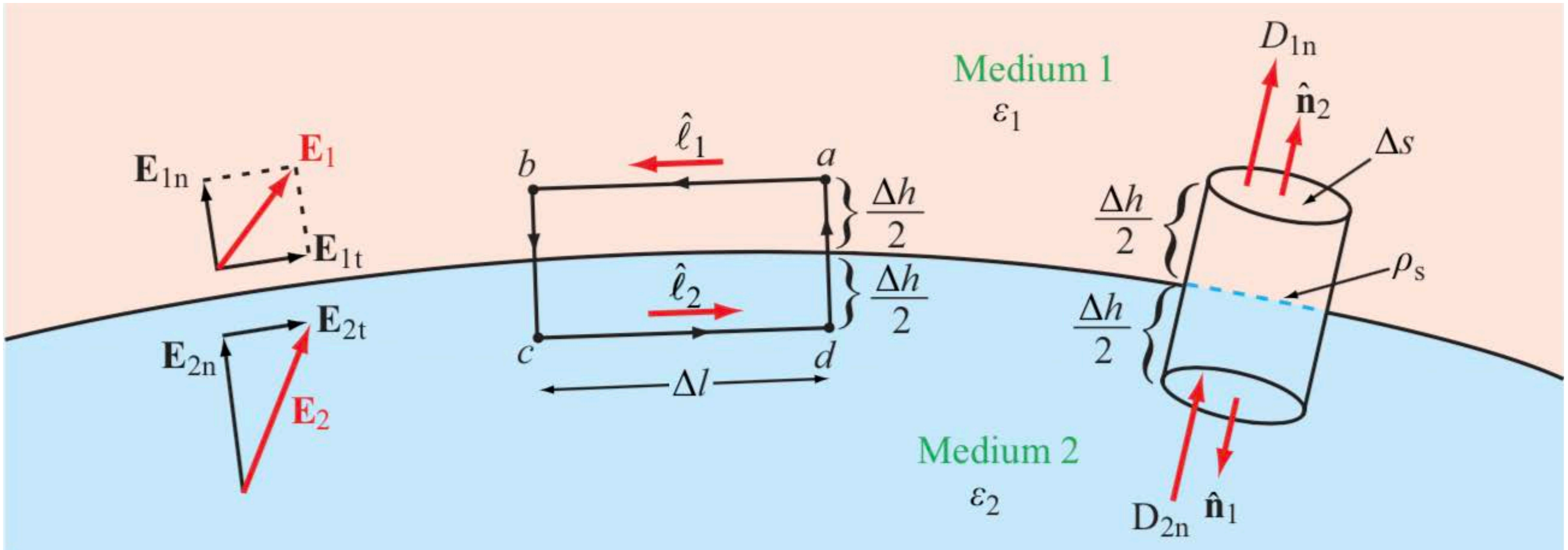
$$\oint_S \vec{D} \cdot d\vec{s} = Q \qquad \vec{\nabla} \cdot \vec{D} = \rho_V$$

Let cylinder height $\Delta h \rightarrow 0$ → only flux is from top/bottom surfaces.

$$Q = \rho_s \Delta s$$ Surface charge density × surface differential

$$\oint_S \vec{D} \cdot d\vec{s} = \int_{top} \vec{D}_1 \cdot \hat{n}_2 ds + \int_{bottom} \vec{D}_2 \cdot \hat{n}_1 ds = \rho_s \Delta s$$

Normal components of $\vec{E}$ and $\vec{D}$



$\hat{n}_1$and $\hat{n}_2$ are <u>outward normal</u> unit vectors from the surface

$$\hat{n}_1 = -\hat{n}_2$$

$$\hat{n}_2 \cdot (\vec{D}_1 - \vec{D}_2) = \rho_s \qquad [\text{C/m}^2]$$

$$\boxed{D_{1n} - D_{2n} = \rho_s} \qquad [\text{C/m}^2]$$

Normal components of $\vec{D}$ change by surface charge density

$$\hat{n}_2 \cdot (\epsilon_1 \vec{E}_1 - \epsilon_2 \vec{E}_2) = \rho_s \qquad \boxed{\epsilon_1 E_{1n} - \epsilon_2 E_{2n} = \rho_s}$$

Summary

Conservative property of $\vec{E}$:

$$\vec{\nabla} \times \vec{E} = 0 \longrightarrow \oint_C \vec{E} \cdot d\vec{l} = 0 \qquad \boxed{\vec{E}_{1t} = \vec{E}_{2t}}$$

Divergence of $\vec{D}$:

$$\vec{\nabla} \cdot \vec{D} = \rho_V \longrightarrow \oint_S \vec{D} \cdot d\vec{s} = Q \qquad \boxed{\epsilon_1 E_{1n} - \epsilon_2 E_{2n} = \rho_s}$$

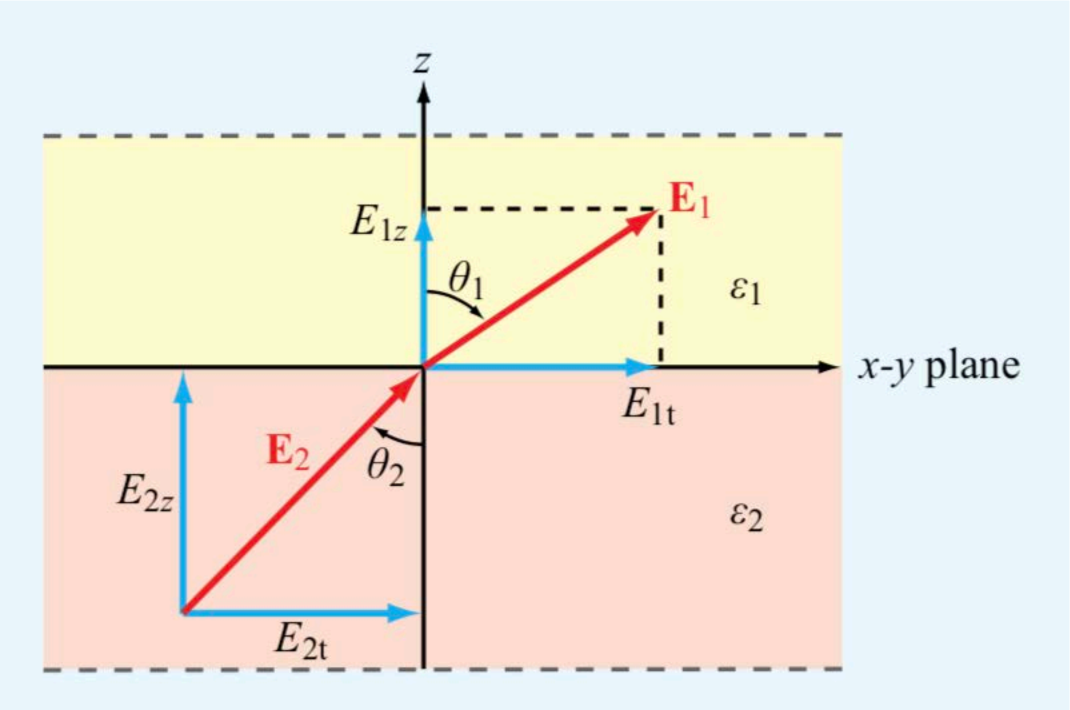
Summary of Boundary Conditions

Table 4-3: Boundary conditions for the electric fields.

Field Component	Any Two Media	Medium 1 Dielectric ε_1	Medium 2 Conductor
Tangential E	$\mathbf{E}_{1t} = \mathbf{E}_{2t}$	$\mathbf{E}_{1t} = \mathbf{E}_{2t} = 0$	
Tangential D	$\mathbf{D}_{1t}/\varepsilon_1 = \mathbf{D}_{2t}/\varepsilon_2$	$\mathbf{D}_{1t} = \mathbf{D}_{2t} = 0$	
Normal E	$\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s$	$E_{1n} = \rho_s/\varepsilon_1$	$E_{2n} = 0$
Normal D	$D_{1n} - D_{2n} = \rho_s$	$D_{1n} = \rho_s$	$D_{2n} = 0$

Notes: (1) ρ_s is the surface charge density at the boundary; (2) normal components of $\mathbf{E}_1$, $\mathbf{D}_1$, $\mathbf{E}_2$, and $\mathbf{D}_2$ are along $\hat{\mathbf{n}}_2$, the outward normal unit vector of medium 2.

Example: 2 dielectrics

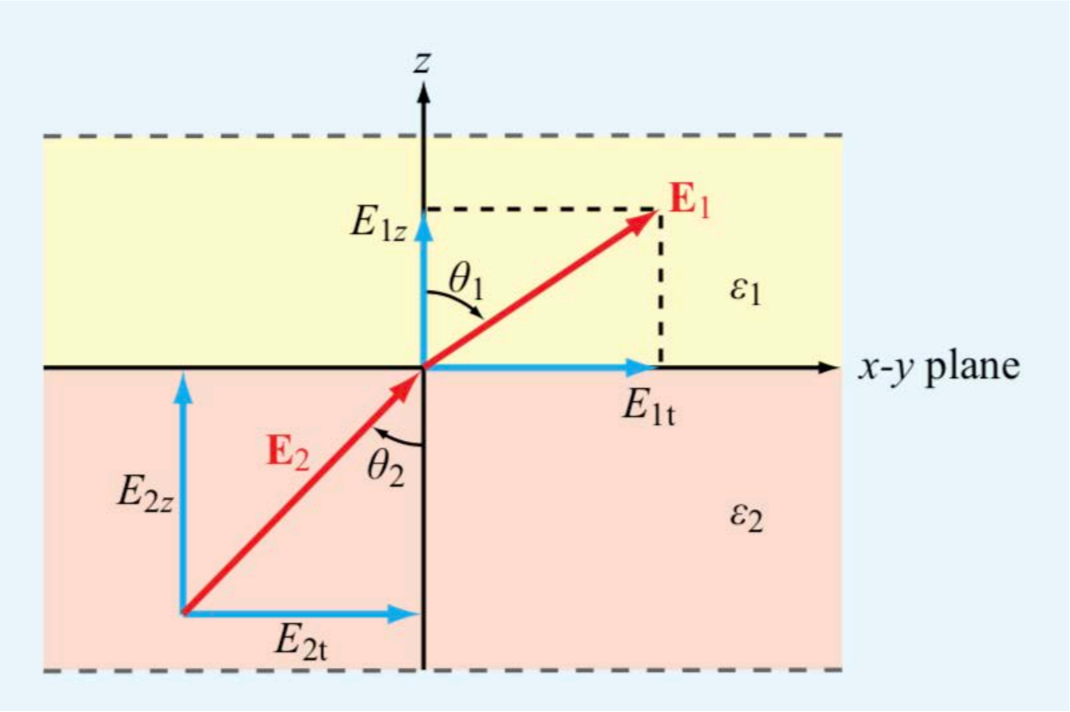


x-y plane boundary charge-free between 2 dielectrics, ϵ_1 and ϵ_2

$\vec{E}_1 = \hat{x}E_{1x} + \hat{y}E_{1y} + \hat{z}E_{1z}$
in dielectric ϵ_1

Find $\vec{E}_2, \theta_1, \theta_2$

Example: 2 dielectrics



x-y plane boundary charge-free between 2 dielectrics, ϵ_1 and ϵ_2

$\vec{E}_1 = \hat{x}E_{1x} + \hat{y}E_{1y} + \hat{z}E_{1z}$
in dielectric ϵ_1

Find $\vec{E}_2, \theta_1, \theta_2$

Let $\vec{E}_2 = \hat{x}E_{2x} + \hat{y}E_{2y} + \hat{z}E_{2z}$ Normal to interface is $\hat{z}$, x-y are tangential

$E_{2x} = E_{1x}$ and $E_{2y} = E_{1y}$ tangential

$D_{2z} = D_{1z} \rightarrow \epsilon_2 E_{2z} = \epsilon_1 E_{1z}$ normal (charge free)

$$\boxed{\vec{E}_2 = \hat{x}E_{1x} + \hat{y}E_{1y} + \hat{z}\frac{\epsilon_1}{\epsilon_2}E_{1z}}$$

Example: 2 dielectrics

To obtain angles:

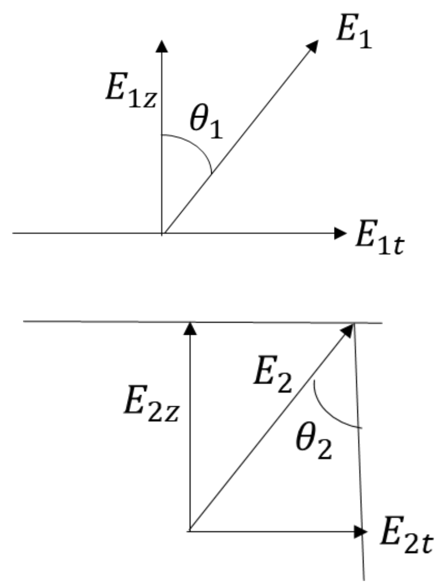
$$E_{1t} = \sqrt{{E_{1x}}^2 + {E_{1y}}^2}$$

$$E_{2t} = \sqrt{{E_{2x}}^2 + {E_{2y}}^2}$$

$$tan\theta_1 = \frac{E_{1t}}{E_{1z}} = \frac{\sqrt{{E_{1x}}^2 + {E_{1y}}^2}}{E_{1z}}$$

$$tan\theta_1 = \frac{E_{1t}}{E_{1z}} = \frac{\sqrt{{E_{1x}}^2 + {E_{1y}}^2}}{E_{1z}}$$

$$tan\theta_2 = \frac{E_{2t}}{E_{2z}} = \frac{\sqrt{{E_{2x}}^2 + {E_{2y}}^2}}{E_{2z}}$$



$$\frac{tan\theta_2}{tan\theta_1} = \frac{\epsilon_2}{\epsilon_1}$$

Dielectric – conductor boundary

Perfect conductor medium 2:

$$\vec{E}_2 = \vec{D}_2 \quad = 0$$

$$E_{1t} = D_{1t} = 0$$

$$D_{1n} = \epsilon_1 E_{1n} \quad = \rho_s$$

$\hat{n}$

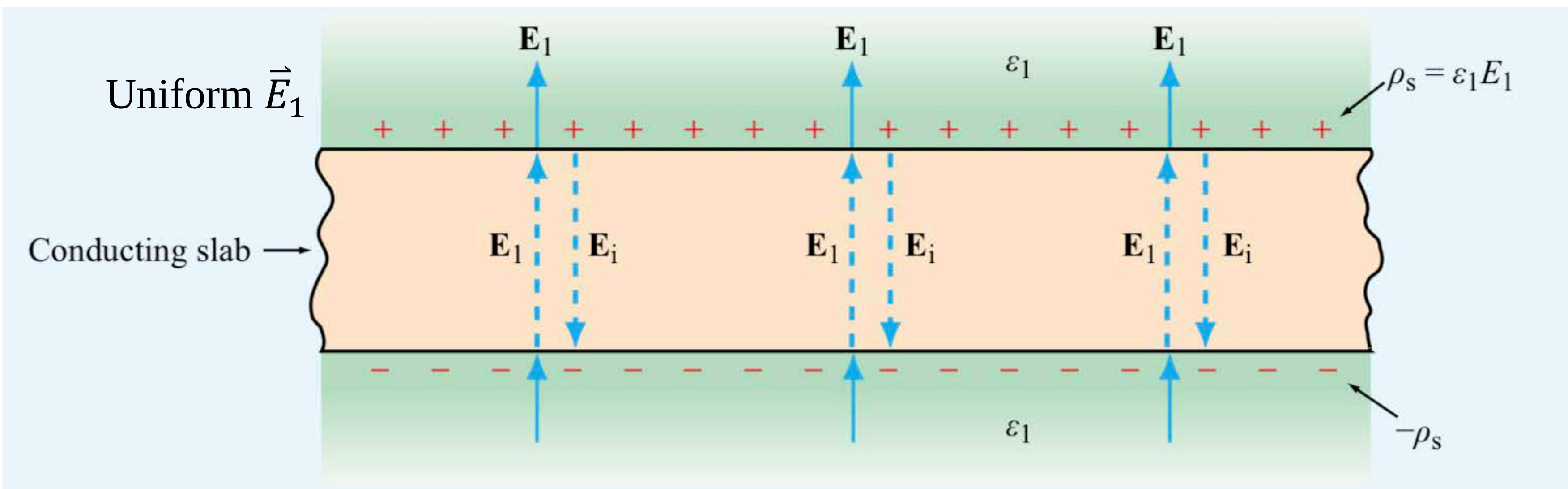
Medium 1: ϵ_1

Medium 2: $\sigma = \infty$

$\hat{n}$ from conductor

$$\boxed{\vec{D}_1 = \epsilon_1 \vec{E}_1 = \hat{n}\rho_s}$$

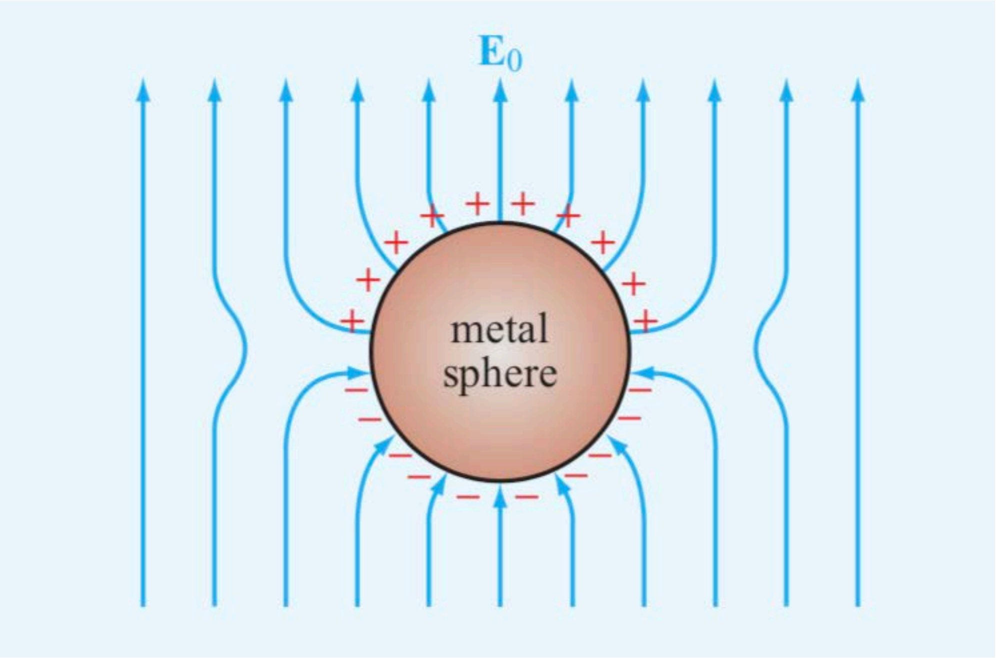
At conductor surface



$\vec{E}_i = -\vec{E}_1$ since field inside conductor must = 0

Net electric field inside a conductor is zero

Dielectric – conductor boundary



Metal in external field, $\vec{E}_0$

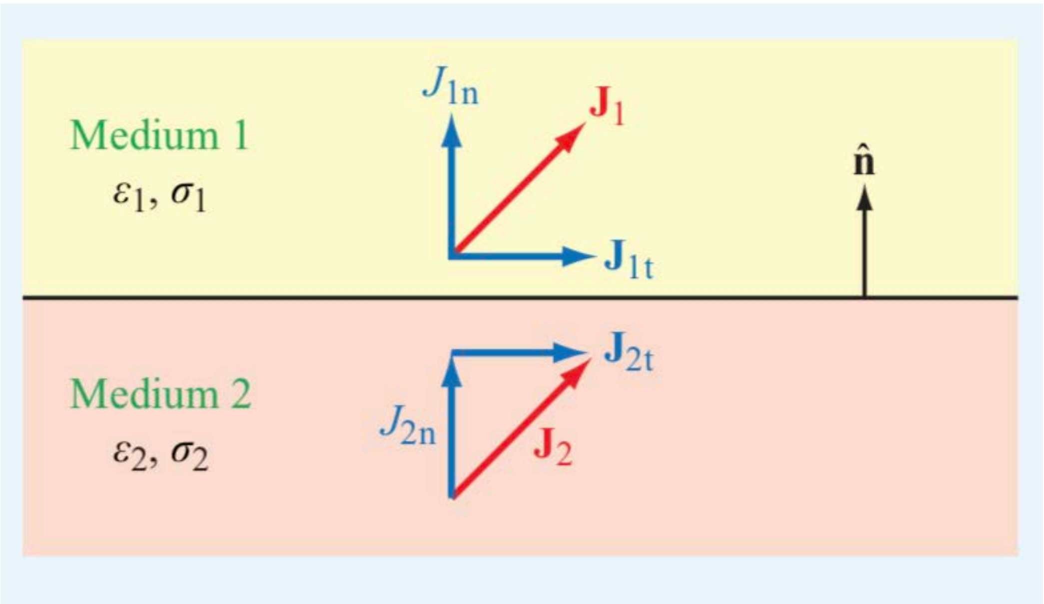
$\vec{E}$ points in = neg charge

$\vec{E}$ points out = pos charge

$\vec{E}$ always normal to conductor

Conductor – conductor boundary

General, not perfect dielectric or perfect conductor



$$\vec{E}_{1t} = \vec{E}_{2t}$$

$$\epsilon_1 E_{1n} - \epsilon_2 E_{2n} = \rho_s$$

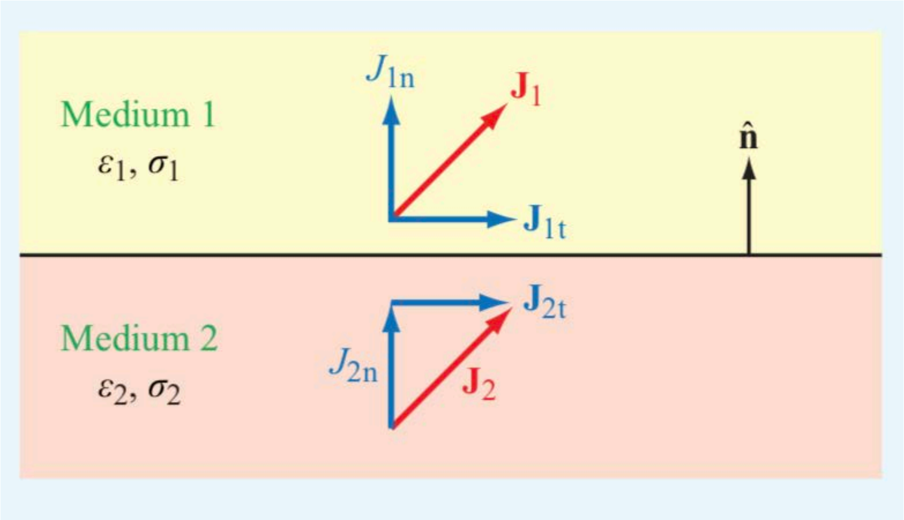
Electric field → gives rise to $\vec{J}_s$

$$\vec{J}_1 = \sigma_1 \vec{E}_1 \qquad \vec{J}_2 = \sigma_2 \vec{E}_2$$

$$E_{1n} = \frac{J_{1n}}{\sigma_1} \qquad E_{2n} = \frac{J_{2n}}{\sigma_2}$$

Conductor – conductor boundary

General, not perfect dielectric or perfect conductor



$$\vec{E}_{1t} = \vec{E}_{2t}$$

$$\epsilon_1 E_{1n} - \epsilon_2 E_{2n} = \rho_s$$

Electric field → gives rise to $\vec{J}_s$

$$\vec{J}_1 = \sigma_1 \vec{E}_1 \qquad \vec{J}_2 = \sigma_2 \vec{E}_2$$

$$E_{1n} = \frac{J_{1n}}{\sigma_1} \qquad E_{2n} = \frac{J_{2n}}{\sigma_2}$$

$$\frac{\vec{J}_{1t}}{\sigma_1} = \frac{\vec{J}_{2t}}{\sigma_2} \qquad \epsilon_1 \left(\frac{J_{1n}}{\sigma_1}\right) - \epsilon_2 \left(\frac{J_{2n}}{\sigma_2}\right) = \rho_s$$

$\vec{J}_{1t}, \vec{J}_{2t}$ → current densities flowing in the 2 media parallel to boundary

If $J_{1n} \neq J_{2n}$ → then we have varying ρ_s at boundary and no longer electrostatics

<u>For electrostatics, $J_{1n} = J_{2n}$</u> $\qquad J_{1n}\left(\frac{\epsilon_1}{\sigma_1} - \frac{\epsilon_2}{\sigma_2}\right) = \rho_s$

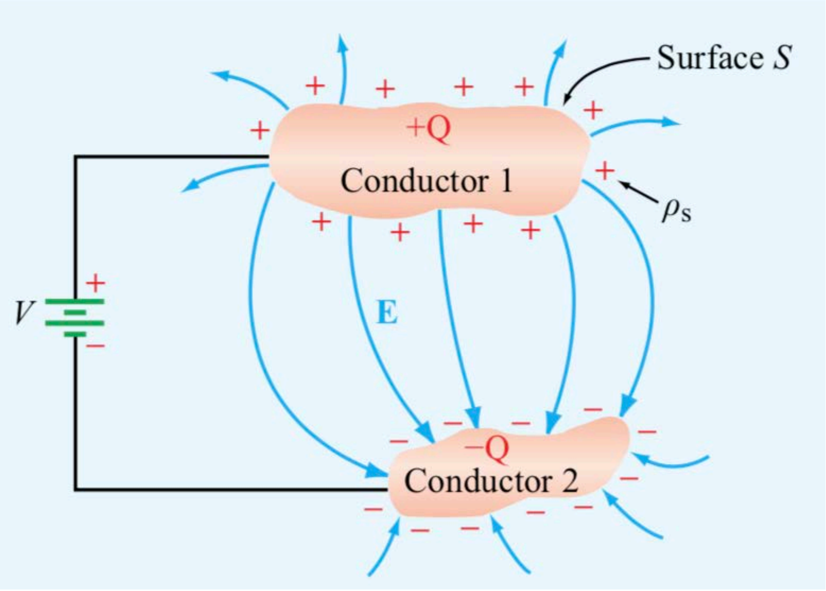
Capacitance

Separate 2 conductors by dielectric → forms a capacitor

Conductor excess charge always at surface distributed to maintain $\vec{E} = 0$ (equipotential) everywhere inside conductor

$$\text{Capacitance} = C = \frac{Q}{V} \qquad [\text{C/V or F}]$$

V: potential / voltage difference between conductors



$\vec{E}$ → always E_n ($E_t = 0$ for conductors)

$$\boxed{E_n = \hat{n} \cdot \vec{E} = \frac{\rho_s}{\epsilon}}$$

Insulator between conductors

At conductor surface

Capacitance

$$Q = \int_S \rho_S ds = \int_S \epsilon \hat{n} \cdot \vec{E} ds = \int_S \epsilon \vec{E} \cdot d\vec{s}$$

$$V = V_{12} = -\int_{P_2}^{P_1} \vec{E} \cdot d\vec{l}$$

P_1, P_2 any points on conductors 1 and 2

$$C = \frac{Q}{V} = \frac{\int_S \epsilon \vec{E} \cdot d\vec{s}}{-\int_l \vec{E} \cdot d\vec{l}}$$

$S = +Q$ surface
P_1 is on S

Integration path from conductor 2 to 1

* Value of C is independent of E. It depends on geometry and materials

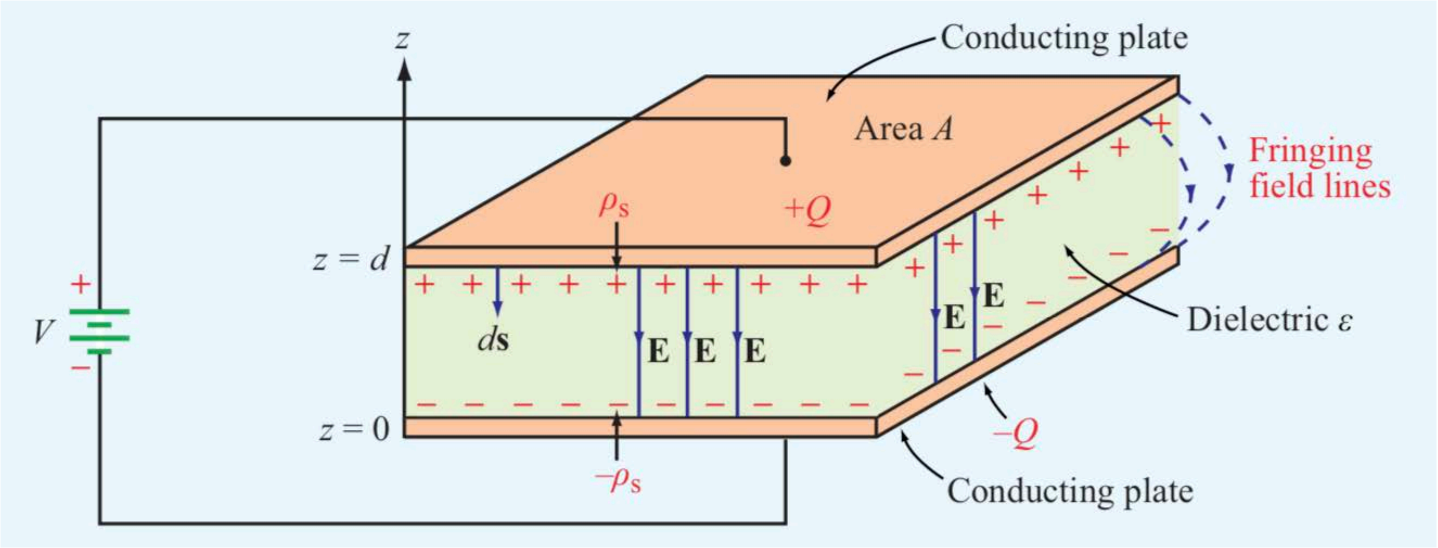
If dielectric is not perfect, can have some conductivity, resistance = R

$$R = \frac{V}{I} = \frac{-\int_l \vec{E} \cdot d\vec{l}}{\int_S \sigma \vec{E} \cdot d\vec{s}}$$

For medium with uniform σ, ϵ: RC = ϵ/σ

Parallel plate capacitor

2 parallel plates, surface area A separated by d filled with dielectric, ϵ.



Ignore fringing fields

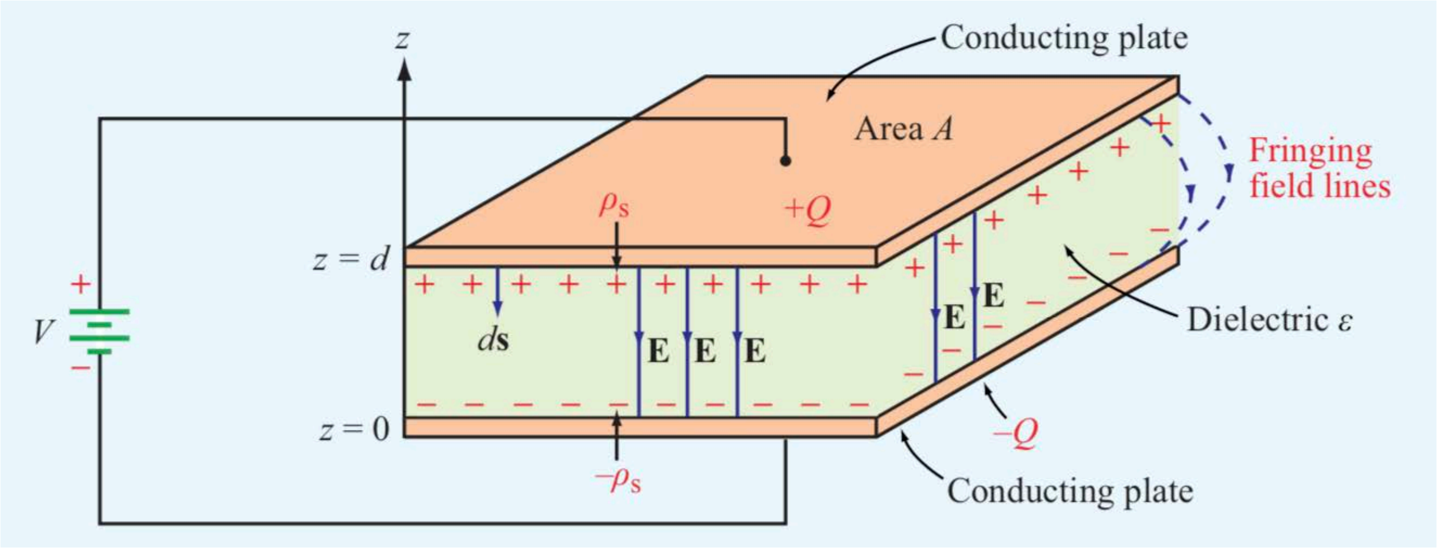
$\rho_s = \frac{Q}{A}$ upper plate

$\vec{E} = -\hat{z}E$ (assume uniform)

$$E_n = \hat{n} \cdot \vec{E} = \frac{\rho_s}{\epsilon} \quad \Rightarrow \quad E_n = \frac{\rho_s}{\epsilon} = \frac{Q}{\epsilon A}$$

Parallel plate capacitor

2 parallel plates, surface area A separated by d filled with dielectric, ϵ.



Ignore fringing fields

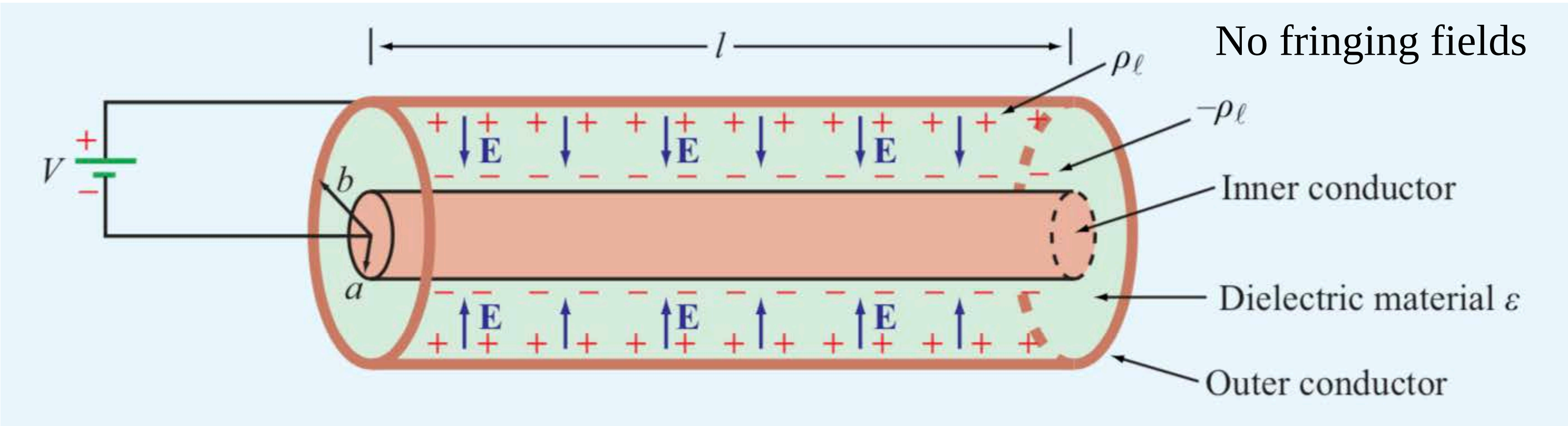
$\rho_s = \frac{Q}{A}$ upper plate $\quad \vec{E} = -\hat{z}E$ (assume uniform) $\quad E_n = \frac{\rho_s}{\epsilon} = \frac{Q}{\epsilon A}$

$$V = -\int_0^d \vec{E} \cdot d\vec{l} = -\int_0^d (-\hat{z}E) \cdot \hat{z}dz = Ed \qquad E = \frac{Q}{\epsilon A}$$

$$C = \frac{Q}{V} = \frac{Q}{Ed} = \frac{\epsilon A}{d}$$

C is independent of E. It depends on geometry and materials

Coax capacitance



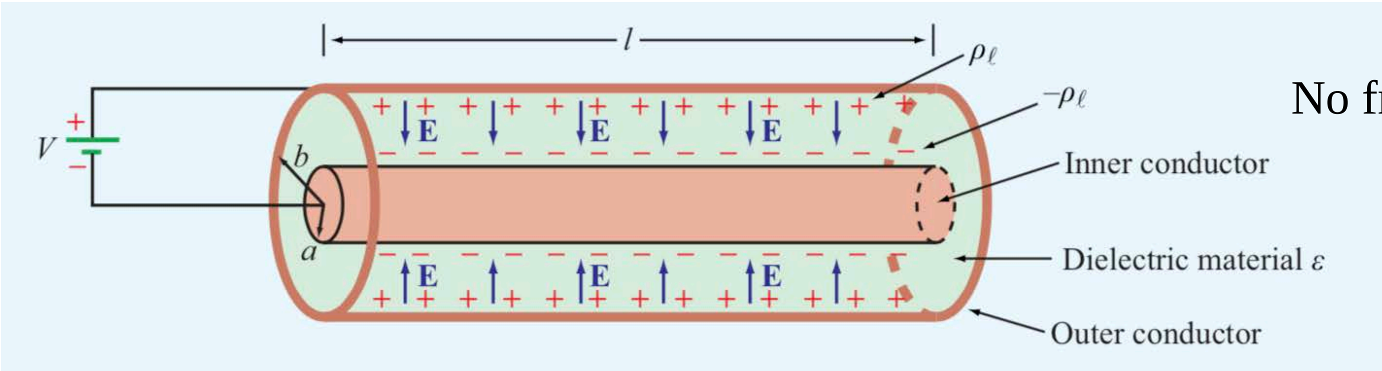
Given voltage, V

+Q accumulates on outer conductor   -Q accumulates on inner conductor

Assume uniform charge distributions: (outer) $\rho_s' = \frac{Q}{2\pi b l}$ (inner) $\rho_s'' = \frac{-Q}{2\pi a l}$

E_n

Coax capacitance



No fringing fields

Cylindrical Gaussian surface: $a < r < b$

$$\oint_S \vec{D} \cdot d\vec{s} = Q$$

$$\oint_S \underbrace{\epsilon\vec{E}}_{E_r} \cdot \underbrace{d\vec{s}}_{2\pi r l} = Q$$

$\vec{E}$ in $\hat{r}$ direction, points inward: $\vec{E} = -\hat{r}\dfrac{Q}{2\pi\epsilon r l}$

Potential between outer and inner:

$$V = -\int_a^b \vec{E} \cdot d\vec{l} = -\int_a^b \left(-\hat{r}\frac{Q}{2\pi\epsilon r l}\right) \cdot (\hat{r}dr) = \frac{Q}{2\pi\epsilon l}\ln\left(\frac{b}{a}\right)$$

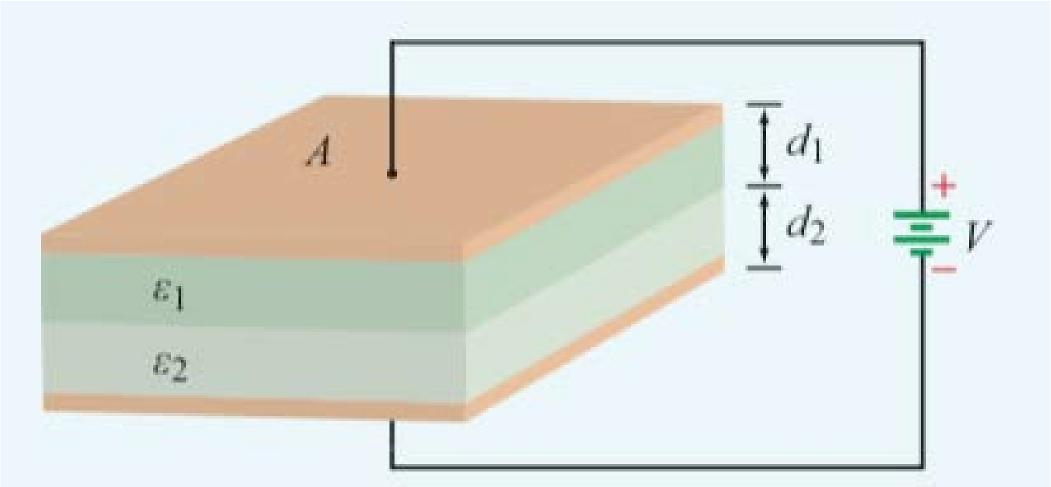
$$C = \frac{Q}{V} = \frac{2\pi\epsilon l}{\ln\left(\frac{b}{a}\right)}$$

<u>Table 2.1</u> → $C' = \frac{C}{l} = \frac{2\pi\epsilon}{\ln\left(\frac{b}{a}\right)}$

E_n

Parallel plate capacitor

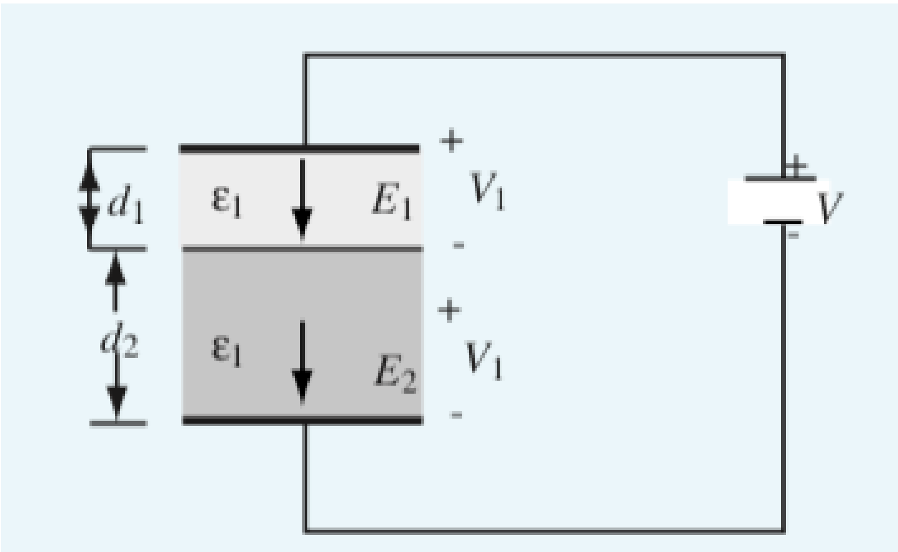
The capacitor shown in the following figure consists of two parallel dielectric layers.



V_1 and V_2 are the electric potentials across the upper and lower dielectrics, respectively.

Obtain: expressions for E_1 and E_2 in terms of ε_1 , ε_2 , V , and the indicated dimensions of the capacitor.

Parallel plate capacitor



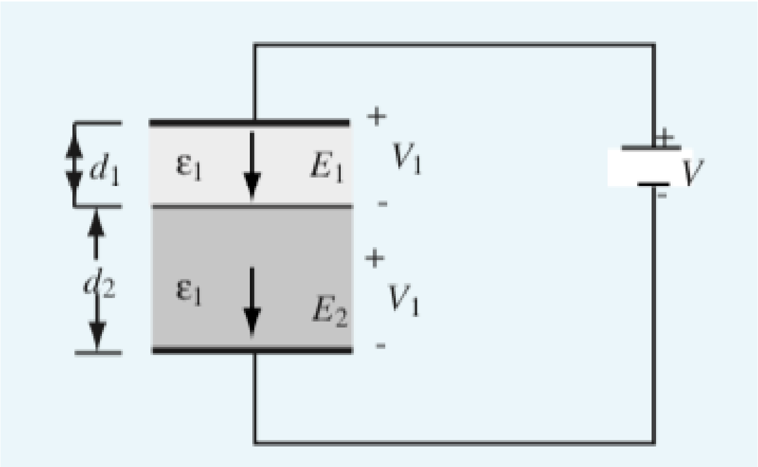
If V_1 is the voltage across the top layer and V_2 across the bottom layer, then:

$$V = V_1 + V_2$$

$$E_1 = \frac{V_1}{d_1} \qquad E_2 = \frac{V_2}{d_2}$$

Boundary conditions → the normal component of $\vec{\boldsymbol{D}}$ is continuous across the boundary (in the absence of surface charge).

Parallel plate capacitor



$$E_1 = \frac{V_1}{d_1}$$

$$E_2 = \frac{V_2}{d_2}$$

Boundary conditions → the normal component of $\vec{\boldsymbol{D}}$ is continuous

Electric fields inside of capacitor:

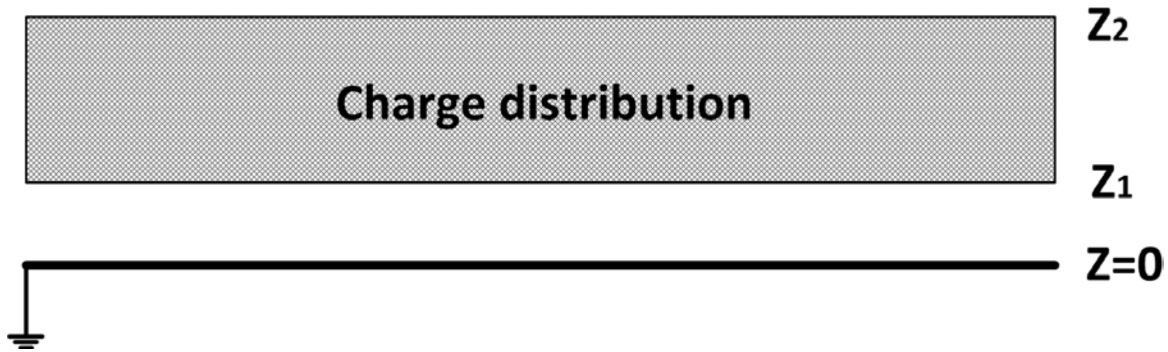
$$D_{1n} = D_{2n} \rightarrow \ \varepsilon_1 E_1 = \varepsilon_2 E_2$$

$$V = V_1 + V_2 = E_1 d_1 + E_2 d_2 = E_1 d_1 + \frac{\varepsilon_1 E_1}{\varepsilon_2} d_2$$

Solve for E_1 and E_2: $\quad E_1 = \dfrac{V}{d_1 + \frac{\varepsilon_1}{\varepsilon_2} d_2} \qquad E_2 = \dfrac{V}{d_2 + \frac{\varepsilon_2}{\varepsilon_1} d_1}$

Capacitor with Uniform Charge Density

Consider a plate placed at $z = 0$ in a nonmagnetic material with dielectric ϵ. The plate has a ground potential. A charge distribution $\rho_v = \rho_0$ is placed from z_1 to z_2 above the plate, as seen in the figure below



There are two regions to consider, $0 < z \leq z_1$ and $z_1 < z \leq z_2$

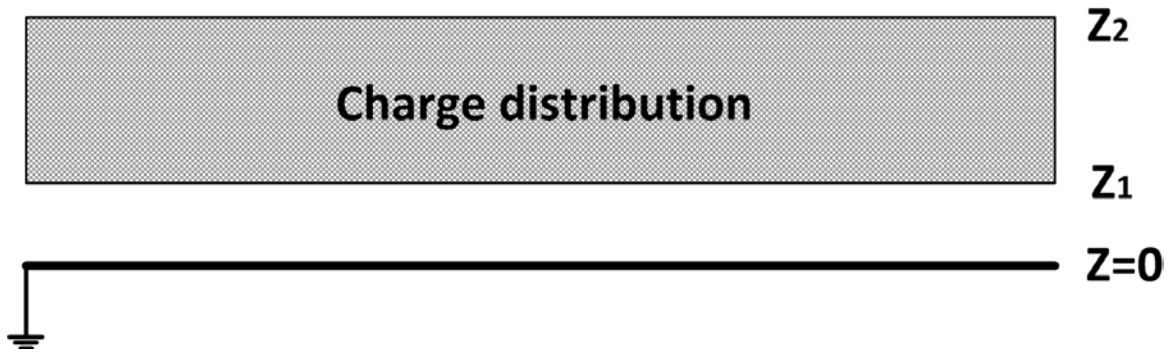
Using Poisson's Equation, below, obtain an expression for the voltage and **E**-field in each region

$$\nabla^2 V = -\frac{\rho_v}{\epsilon}$$

Capacitor with Uniform Charge Density

First, write Poisson's Equation as the Laplacian of the scalar function V

$$\nabla^2 V = \nabla \cdot (\nabla V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$



Clearly for this geometry V is exclusively a function of z, therefore

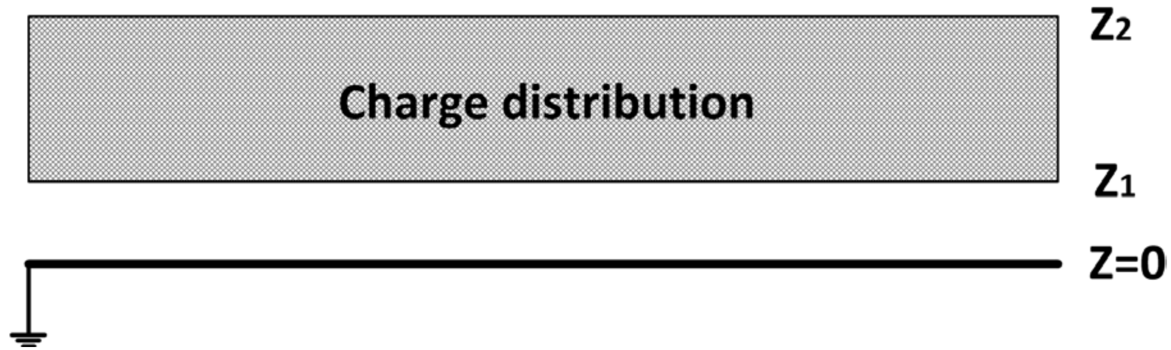
$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 V}{\partial y^2} = 0$$

$$\frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\epsilon}$$

Capacitor with Uniform Charge Density

There is no charge distribution in our first region, $0 < z \leq z_1$ so Poisson's equation simplifies to Laplace's

$$\frac{\partial^2 V}{\partial z^2} = 0$$



Thus, the solution is of the form

$$V(0 < z \leq z_1) = Az + B$$

Applying the known boundary condition that $V(z = 0) = 0 \rightarrow B = 0$

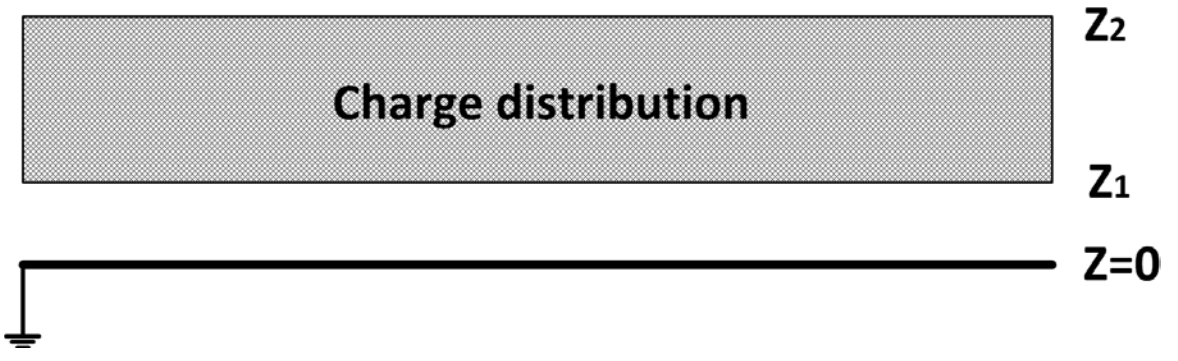
$$V(0 < z \leq z_1) = Az$$

Where A is a currently unknown constant

Capacitor with Uniform Charge Density

For our second region, $z_1 < z \leq z_2$

$$\frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\epsilon}$$



Thus, the solution is of the form

$$V(z_1 < z \leq z_2) = Bz^2 + Cz + D$$

We can solve for one of our unknowns using our Poisson's Equation

$$\frac{\partial V(z_1 < z \leq z_2)}{\partial z} = 2Bz + C$$

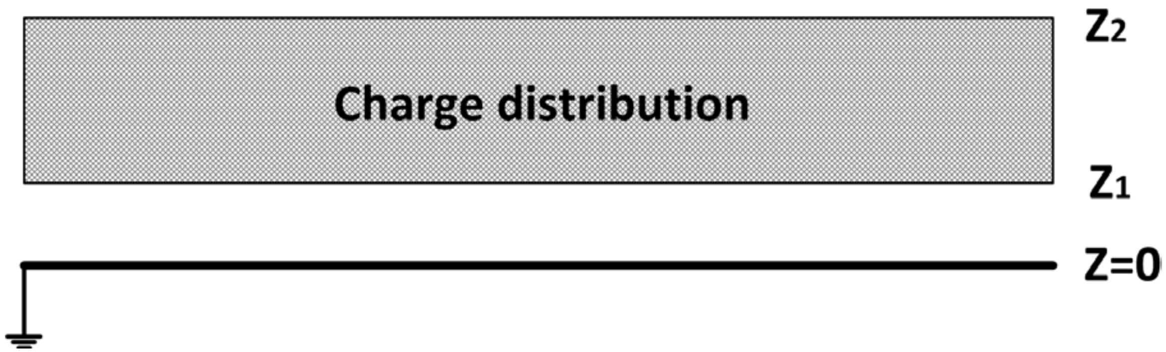
$$\frac{\partial^2 V(z_1 < z \leq z_2)}{\partial z^2} = 2B = -\frac{\rho_v}{\epsilon} \rightarrow B = -\frac{\rho_v}{2\epsilon}$$

$$V(z_1 < z \leq z_2) = -\frac{\rho_v}{2\epsilon} z^2 + Cz + D$$

Capacitor with Uniform Charge Density

We have

$$V(0 < z \leq z_1) = Az$$
$$V(\mathrm{z}_1 < z \leq z_2) = -\frac{\rho_v}{2\epsilon} z^2 + Cz + D$$



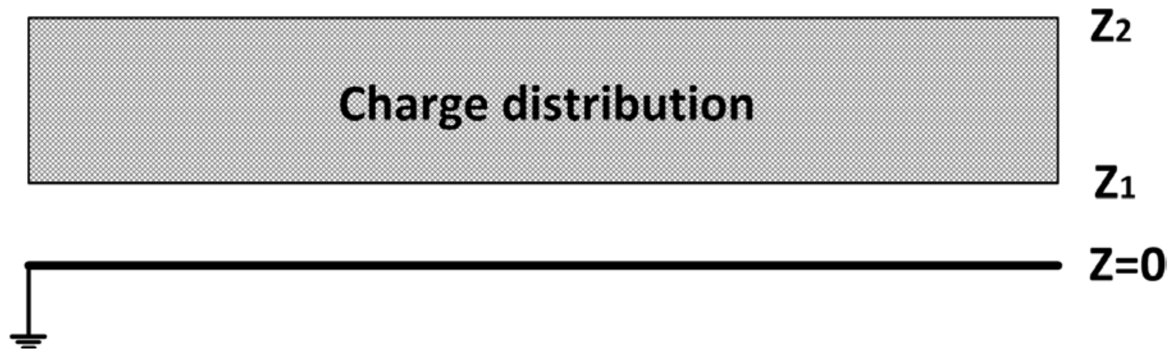
From here we use the relation $E = -\nabla V \rightarrow E(z) = -\frac{\partial V}{\partial \mathrm{z}}$

$$E(0 < z \leq z_1) = -A$$
$$E(\mathrm{z}_1 < z \leq z_2) = \frac{\rho_v}{\epsilon} z - C$$

Capacitor with Uniform Charge Density

We have

$$E(0 < z \leq z_1) = -A$$
$$E(\mathrm{z}_1 < z \leq z_2) = \frac{\rho_v}{\epsilon} z - C$$



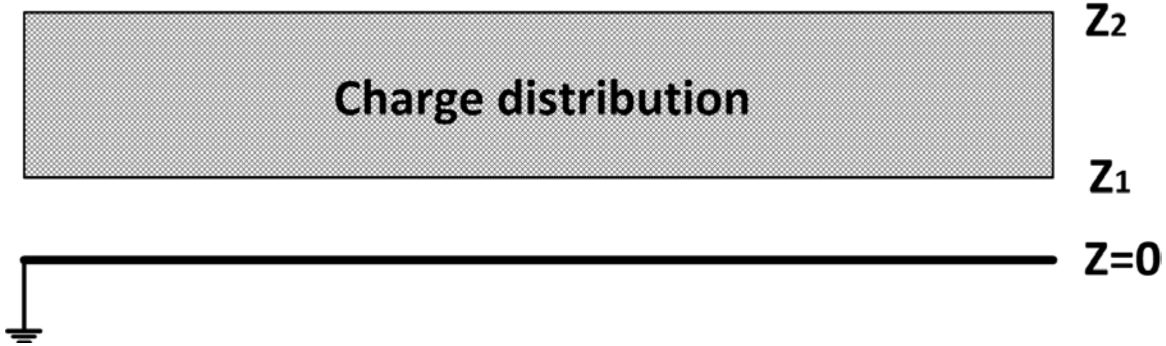
We know that $V(z)$ at the interfaces between regions and because ϵ is continuous we know the $E(z)$ is also continuous at these interfaces.

We also know that the $E(z)$ goes to 0 outside the plates ($z > z_2$) as the opposite charge is accumulated on the ground plate relative to the opposing charge distribution ($E(z > z_2) = 0$)

Capacitor with Uniform Charge Density

Applying all boundary conditions

$$Az_1 = -\frac{\rho_v}{2\epsilon} z_1^2 + Cz_1 + D$$
$$-A = \frac{\rho_v}{\epsilon} z_1 - C$$
$$0 = \frac{\rho_v}{\epsilon} z_2 - C$$



With three unknowns (A, C, and D) and three equations, the rest is linear algebra

Capacitor with Uniform Charge Density

Applying all boundary conditions

$$\mathrm{C} = \frac{\rho_v}{\epsilon} z_2$$

$$A = \frac{\rho_v}{\epsilon} z_2 - \frac{\rho_v}{\epsilon} z_1 = \frac{\rho_v}{\epsilon}(z_2 - z_1)$$

$$D = \frac{\rho_v}{2\epsilon} z_1^2 + (A - C) z_1 = \frac{\rho_v}{2\epsilon} z_1^2 + \left(\frac{\rho_v}{\epsilon} z_2 - \frac{\rho_v}{\epsilon} z_1 - \frac{\rho_v}{\epsilon} z_2\right) z_1 = \frac{\rho_v}{2\epsilon} z_1^2 - \frac{\rho_v}{\epsilon} z_1^2 = -\frac{\rho_v}{\epsilon} z_1^2$$

Plugging these values in to our equations for $V(z)$ and $E(z)$

$$V(0 < z \leq z_1) = \frac{\rho_v}{\epsilon}(z_2 - z_1) z$$

$$V(\mathrm{z}_1 < z \leq z_2) = -\frac{\rho_v}{2\epsilon} z^2 + \frac{\rho_v}{\epsilon} z_2 z - \frac{\rho_v}{\epsilon} z_1^2 = -\frac{\rho_v}{\epsilon}(z^2 - z_2 z + z_1^2)$$

$$E(0 < z \leq z_1) = -\frac{\rho_v}{\epsilon}(z_2 - z_1) = \frac{\rho_v}{\epsilon}(z_1 - z_2)$$

$$E(\mathrm{z}_1 < z \leq z_2) = \frac{\rho_v}{\epsilon} z - \frac{\rho_v}{\epsilon} z_2 = \frac{\rho_v}{\epsilon}(z - z_2)$$

Electrostatic Potential Energy

Electrostatic potential energy density (Joules/volume)

$$w_e = \frac{W_e}{\mathcal{V}} = \frac{1}{2}\,\varepsilon E^2 \qquad (\mathrm{J/m^3}).$$

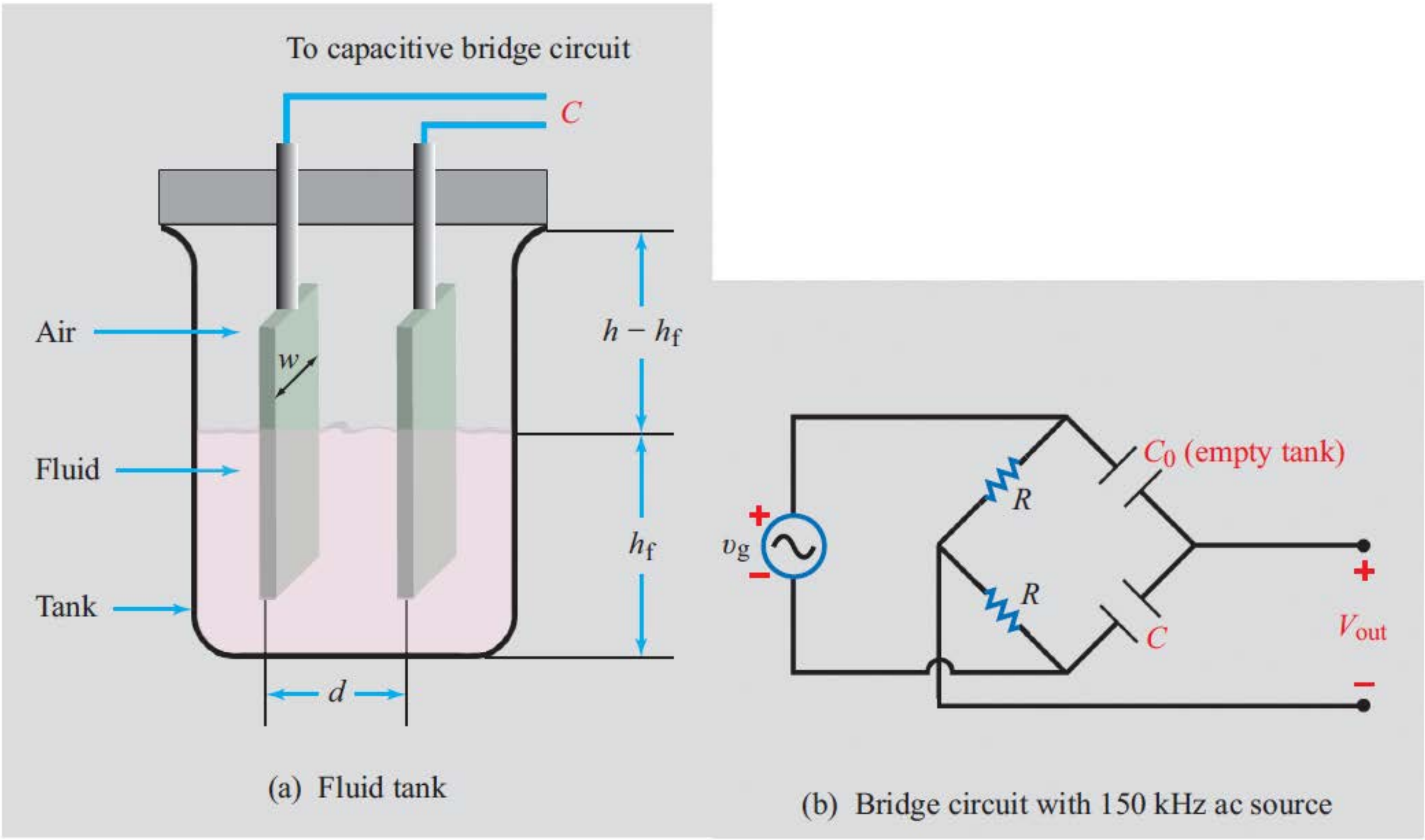
Energy stored in a capacitor

$$W_e = \tfrac{1}{2} C V^2 \qquad (\mathrm{J}).$$

Total electrostatic energy stored in a volume

$$W_e = \frac{1}{2}\int_{\mathcal{V}} \varepsilon E^2 \, d\mathcal{V} \qquad (\mathrm{J})$$

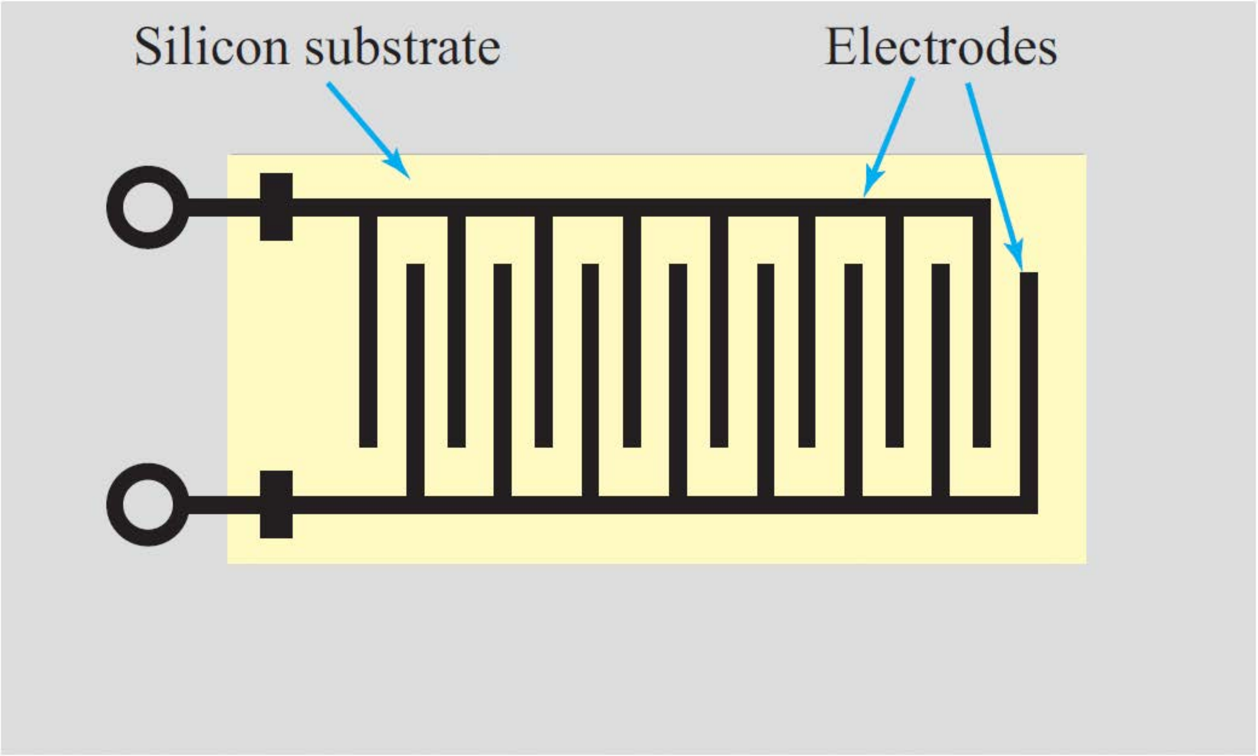
Capacitive Sensors



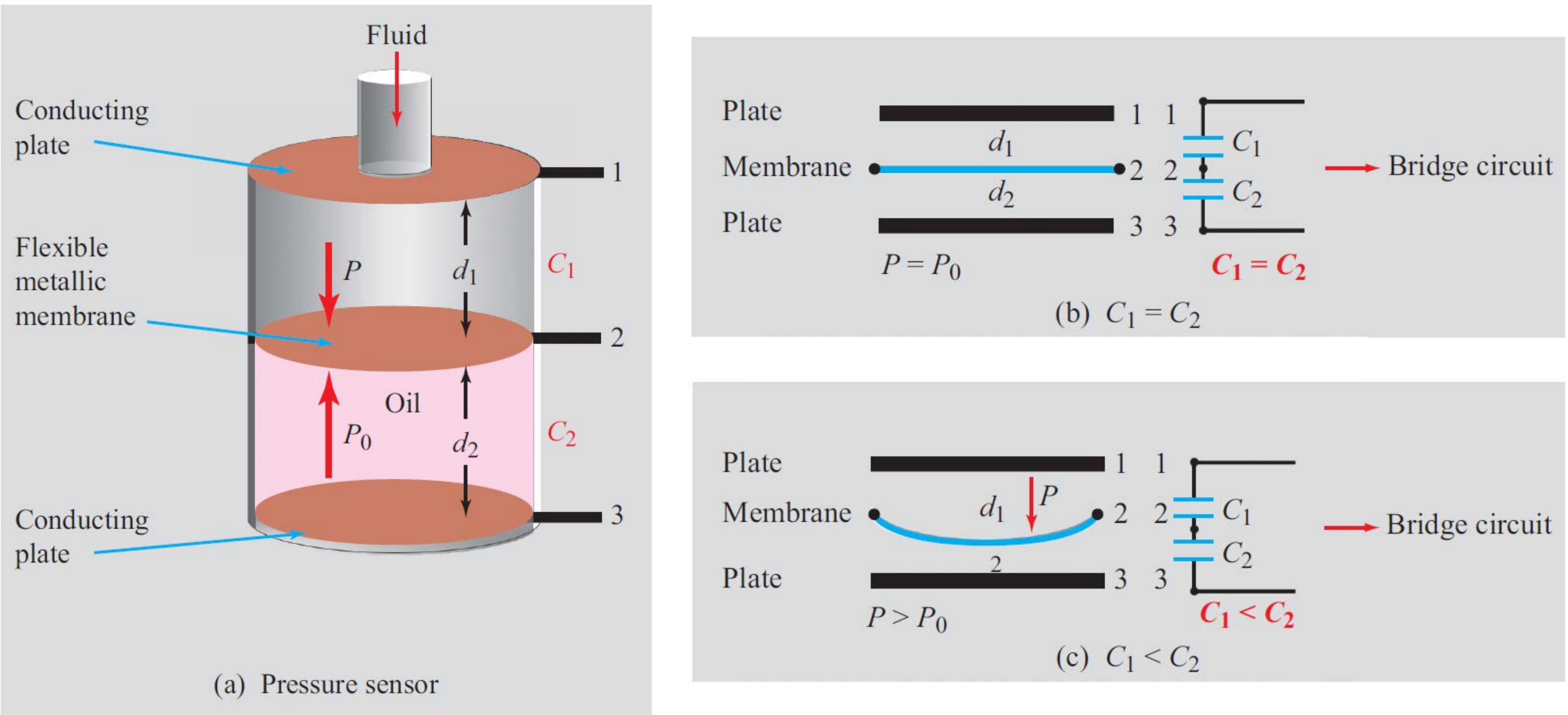
(a) Fluid tank

(b) Bridge circuit with 150 kHz ac source

Humidity Sensor



Pressure Sensor

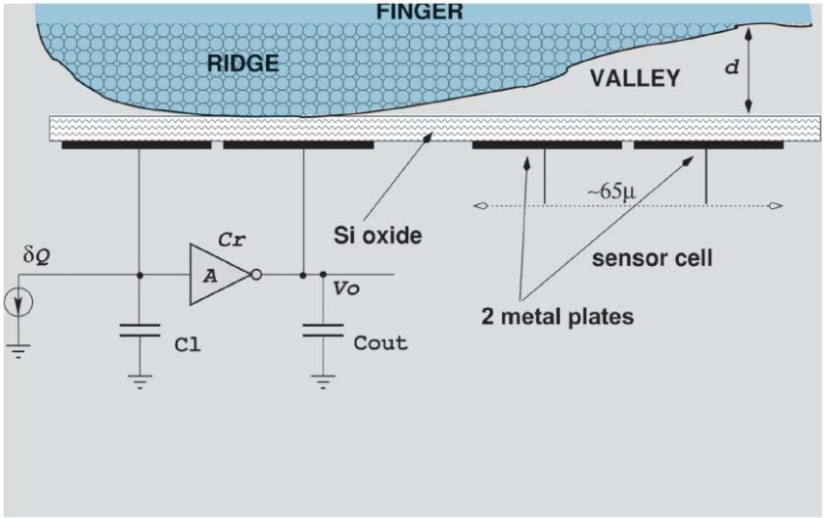


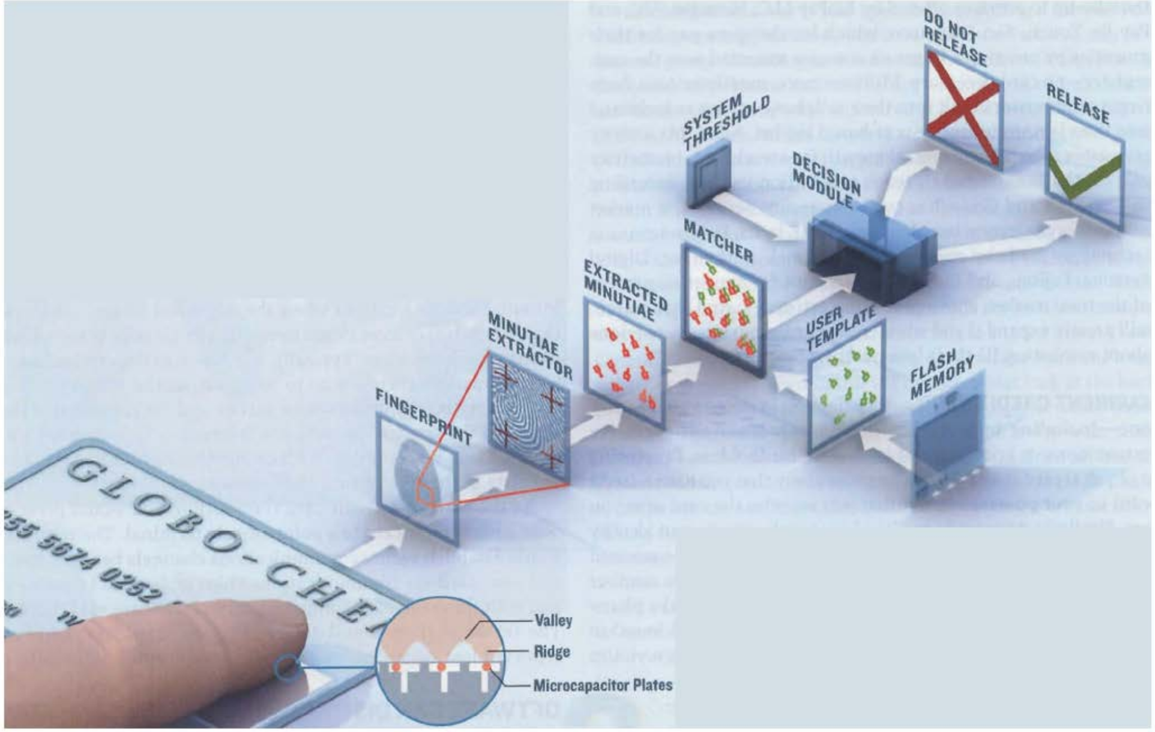
(a) Pressure sensor

(b) $C_1 = C_2$

(c) $C_1 < C_2$

Fingerprint Imager





Maxwell's Equations for Electrostatics

Name	Differential Form	Integral Form
Gauss's law	$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$
Kirchhoff's law	$\nabla \times \mathbf{E} = 0$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$

Current density	$\mathbf{J} = \rho_v \mathbf{u}$
Poisson's equation	$\nabla^2 V = -\dfrac{\rho_v}{\varepsilon}$
Laplace's equation	$\nabla^2 V = 0$
Resistance	$R = \dfrac{-\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_S \sigma \mathbf{E} \cdot d\mathbf{s}}$
Boundary conditions	Table 4-3
Capacitance	$C = \dfrac{\int_S \varepsilon \mathbf{E} \cdot d\mathbf{s}}{-\int_l \mathbf{E} \cdot d\mathbf{l}}$
RC relation	$RC = \dfrac{\varepsilon}{\sigma}$
Energy density	$w_e = \frac{1}{2} \varepsilon E^2$

Electric Field

Point charge	$\mathbf{E} = \hat{\mathbf{R}} \dfrac{q}{4\pi\varepsilon R^2}$
Many point charges	$\mathbf{E} = \dfrac{1}{4\pi\varepsilon} \sum_{i=1}^{N} \dfrac{q_i (\mathbf{R} - \mathbf{R}_i)}{\lvert\mathbf{R} - \mathbf{R}_i\rvert^3}$
Volume distribution	$\mathbf{E} = \dfrac{1}{4\pi\varepsilon} \int_{\mathcal{V}'} \hat{\mathbf{R}}' \dfrac{\rho_v \, d\mathcal{V}'}{R'^2}$
Surface distribution	$\mathbf{E} = \dfrac{1}{4\pi\varepsilon} \int_{S'} \hat{\mathbf{R}}' \dfrac{\rho_s \, ds'}{R'^2}$
Line distribution	$\mathbf{E} = \dfrac{1}{4\pi\varepsilon} \int_{l'} \hat{\mathbf{R}}' \dfrac{\rho_\ell \, dl'}{R'^2}$
Infinite sheet of charge	$\mathbf{E} = \hat{\mathbf{z}} \dfrac{\rho_s}{2\varepsilon_0}$
Infinite line of charge	$\mathbf{E} = \dfrac{\mathbf{D}}{\varepsilon_0} = \hat{\mathbf{r}} \dfrac{D_r}{\varepsilon_0} = \hat{\mathbf{r}} \dfrac{\rho_\ell}{2\pi\varepsilon_0 r}$
Dipole	$\mathbf{E} = \dfrac{qd}{4\pi\varepsilon_0 R^3} (\hat{\mathbf{R}} \, 2\cos\theta + \hat{\boldsymbol{\theta}} \sin\theta)$
Relation to V	$\mathbf{E} = -\nabla V$